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Spectral Properties of a Piecewise Linear Intermittent Map

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For a piecewise linear intermittent map, the evolution of statistical averages of a class of observables with respect to piecewise constant initial densities is investigated and generalized eigenfunctions of the Frobenius–Perron operator $\hat{P}$ are explicitly derived. The evolution of the averages are shown to be a superposition of the contributions from two simple eigenvalues 1 and $\lambda_d \in (-1, 0)$, and a continuous spectrum on the unit interval $[0, 1]$ of $\hat{P}$. Power-law decay of correlations are controlled by the continuous spectrum. Also the non-normalizable invariant measure in the non-stationary regime is shown to determine the strength of the power-law decay.

KEY WORDS: Generalized spectral decomposition; generalized eigenfunctions; intermittent map; non-normalizable invariant measure.

1. INTRODUCTION

The decay of correlation is one of the important characteristic properties of dynamical systems. For hyperbolic systems where all trajectories are exponentially unstable in both directions of time, correlation functions decay exponentially. The decay rates, known as Pollicott–Ruelle resonances,[1, 2] are the logarithms of the zeros of the dynamical zeta function, which is essentially the Fredholm determinant of the Frobenius–Perron operator. Hence, the exponential decay of correlation implies that the resolvent of the Frobenius–Perron operator is meromorphic in an annulus $\{z \mid r_0 < |z| < 1\}$

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with some $0 < r_0 < 1$. The Frobenius–Perron operator may have only discrete eigenvalues there.

This is not the case for non-hyperbolic systems. An important class of non-hyperbolic systems is that of intermittent maps, which and their periodic extensions[3–10] have been intensively studied last decades. For intermittent maps, the correlation decays polynomially and the power-spectra may exhibit polynomial or logarithmic divergence at low frequencies.[3–8] In their periodic extensions,[9, 10] the slow decay of correlation causes anomalous diffusion of Lévy-flight type. According to this behavior, as pointed out by Artuso,[8] the dynamical zeta function of an intermittent map has a branch point at 1 and a cut along the real axis. For a piecewise linear version of the Pomeau–Manneville map[3] introduced by Gaspard and Wang,[6] Hasegawa and Luschei[7] studied the spectral decomposition of the Frobenius–Perron operator. However, its spectral structure is not fully understood. For example, the existence of a cut along the real axis implies the existence of the real continuous spectrum, but the explicit expressions of the corresponding generalized eigenfunctions were not obtained.

Besides the above-mentioned aspects, there is an interesting point concerning the role of invariant measures. Consider an intermittent map having a parameter which controls the statistical behavior of the map. For parameter values corresponding to the stationary stochastic process, the map admits a normalizable invariant measure.[4, 6] And, for parameter values corresponding to the non-stationary stochastic process, it admits a non-normalizable invariant measure and invariant δ-function measures located at marginal fixed points.[4, 6] The normalizable invariant measure certainly corresponds to an eigenfunction of the Frobenius–Perron operator with eigenvalue 1. However, the role of non-normalizable measure in the spectral properties of the Frobenius–Perron operator is not studied so far.

In this paper, as a first step towards the full spectral characterization of the statistical behavior of intermittent maps, we study the evolution of the statistical averages of a class of observables with respect to piecewise constant initial densities for a piecewise linear approximation of Pomeau–Manneville map discussed by Artuso.[8] The averages are expressed as a superposition of an invariant, exponentially decaying and polynomially decaying terms, which are regarded as contributions, respectively, from the eigenvalues 1, λ_d ($-1 < \lambda_d < 0$) and the continuous spectrum on the interval $[0, 1]$ of the Frobenius–Perron operator. The corresponding generalized eigenfunctions are explicitly constructed as well.

When the map is in the stationary regime, the map admits an invariant measure with non-zero density with respect to the Lebesgue measure. The invariant density is proportional to an invariant function f_1, which exists for all parameter values and depends smoothly on the parameter. When the map is in non-stationary regime, the invariant function f_1 defines a non-normalizable measure. We show that this non-normalizable measure appears as a generalized eigenfunction corresponding to the edge value 1 of the continuous spectrum and that the normalizable invariant measure in the stationary regime smoothly changes to a δ-function measure located at the marginal fixed point as the map changes from the stationary to the non-stationary regimes.

The paper is organized as follows. In Section 2, we describe a model and classes of observables and initial densities. Evolution of statistical averages of observables is investigated in Section 3 and generalized eigenfunctions of the Frobenius–Perron operator are derived. Long time behavior and the relation between normalizable and non-normalizable invariant measures are investigated in Section 4. Conclusions are drawn in Section 5.

2. PIECEWISE LINEAR POMEAU–MANNEVILLE MAP

One of the simplest intermittent maps is that introduced by Pomeau and Manneville[3] and its piecewise linear approximation was studied by Gaspard and Wang.[6] Here we consider the piecewise linear version ϕ discussed by Artuso.[8] Those maps have a marginal fixed point at the origin, which produces intermittent chaos. The map ϕ: $[0, 1) \to [0, 1)$ is defined as follows:

$$\phi(x) = \begin{cases} \eta_k(x-\xi_k)+\xi_{k-1}, & (\xi_k \leqslant x \leqslant \xi_{k-1}) \\ \dfrac{x-a}{1-a}, & (a \leqslant x \leqslant 1) \end{cases} \tag{1}$$

where $k = 1, 2,\ldots$ and the variables ξ_k and η_k are given by

$$\xi_{k-1} - \xi_k = \frac{a}{\zeta(\beta)}\left(\frac{1}{k}\right)^{\beta} \qquad (k = 1, 2,\ldots), \quad \xi_0 = a \tag{2}$$

$$\eta_k = \begin{cases} \dfrac{\xi_{k-2}-\xi_{k-1}}{\xi_{k-1}-\xi_k} = \left(\dfrac{k}{k-1}\right)^{\beta}, & (k = 2, 3,\ldots) \\ \dfrac{1-a}{a-\xi_1} = \dfrac{1-a}{a}\zeta(\beta), & (k = 1) \end{cases} \tag{3}$$

where $\beta > 1$ is a parameter and $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ is the Riemann zeta function. The map ϕ has a marginal fixed point $x = 0$ where the map is approximated as $\phi(x) \simeq x + c_0 x^{\beta/(\beta-1)}$ with a positive constant c_0. When

$\beta > 2$, the map is in the stationary regime and, when $\beta \leqslant 2$, it is in the non-stationary regime. Moreover, fluctuations in the stationary regime are normal when $\beta > 3$ and of Lévy-type when $3 \geqslant \beta > 2$.[6]

The Frobenius–Perron operator $\hat{P}$ and its adjoint $\hat{P}^*$ are defined respectively as

$$\hat{P}\rho(x) = \int_0^1 dy\, \delta(x - \phi(y))\, \rho(y) \tag{4}$$

$$\hat{P}^* A(x) = A(\phi(x)). \tag{5}$$

Then, the average value of an observable A at time t with respect to an initial density ρ is given by $(A, \hat{P}^t \rho) \equiv (\hat{P}^{*t} A, \rho)$, where (A, ρ) is the inner product:

$$(A, \rho) = \int_0^1 dx\, A(x)\, \rho(x). \tag{6}$$

In order to extract information about generalized eigenfunctions of the Frobenius–Perron operator, we consider the time evolution of the average values of certain observables with respect to a class of initial densities.

Observables $A(x)$ are assumed to be smooth at the origin in the sense that they behave near $x = 0$ as $a_0 + a_1 x + \mathrm{O}(x^{\beta/(\beta-1)})$ where two constants a_0 and a_1 correspond, respectively, to $A(0)$ and $A'(0)$. More precisely, $A(x)$ is a bounded function such that there exist constants a_0 and a_1, and an inequality

$$|A(x) - a_0 - a_1 x| \leqslant K\, |x|^{\beta/(\beta-1)} \qquad (0 \leqslant x \leqslant 1) \tag{7}$$

holds, where K is a positive constant. Let X_O be a set of such functions, then it is a Banach space with respect to the norm:

$$\|A\|_O = |a_0| + |a_1| + \sup_x |A(x)| + \sup_x \frac{|A(x) - a_0 - a_1 x|}{|x|^{\beta/(\beta-1)}}, \tag{8}$$

and is invariant with respect to the adjoint of the Frobenius–Perron operator $\hat{P}^*$: namely, if $A \in X_O$, then $\hat{P}^* A \in X_O$. Note that since X_O contains a space $C^2[0, 1]$ of twice continuously differentiable functions and $C^2[0, 1]$ is dense in the Hilbert space $L^2[0, 1]$ of square integrable functions, X_O is dense in $L^2[0, 1]$.

For expanding maps such as $x_{t+1} = 2x_t \pmod 1$ where all trajectories are exponentially unstable, the spectrum of the Frobenius–Perron operator consists of points and may contain a certain closed disk depending on the class of initial densities,[1,2,11] and when initial densities are smooth enough such as entire functions of exponential type or polynomials, their evolution is fully characterized by the point spectra of the Frobenius–Perron operator.[11] A similar situation is expected for non-hyperbolic maps. However, as the general situation corresponding to the results by Pollicott[1] and Ruelle[2] is not known, we restrict ourselves to a narrow class of piecewise constant initial densities in order to derive explicit expressions of the generalized eigenfunctions of the Frobenius–Perron operator. More precisely, we consider a set X_D of initial densities ρ given by

$$\rho(x) = \begin{cases} \tilde{\rho}_k, & (\xi_k \leqslant x \leqslant \xi_{k-1};\, k = 1, 2,...) \\ \tilde{\rho}_0, & (a \leqslant x \leqslant 1) \end{cases} \tag{9}$$

where

$$\tilde{\rho}_k = \sum_{l=0}^{+\infty} \frac{\rho_l}{k^{(\beta-1)\,l}} \qquad \text{with} \quad \sum_{l=0}^{+\infty} |\rho_l|\, \theta^l < +\infty \tag{10}$$

for some constant $\theta > 1$. Roughly speaking, ρ_l in (10) corresponds to the lth derivative of the density $\rho(x)$ at the origin and the relation (10) implies the smoothness of the densities at the origin. The space X_D is a Banach space with respect to the norm

$$\|\rho\|_D = \sum_{l=0}^{+\infty} |\rho_l|\, \theta^l, \tag{11}$$

and is invariant under the action of the Frobenius–Perron operator $\hat{P}$: namely, if $\rho \in X_D$, then $\hat{P}\rho \in X_D$. Note that, in contrast to the space X_O of observables, the space X_D of densities is not dense in the Hilbert space $L^2[0, 1]$ of square integrable functions.

3. EVOLUTION OF STATISTICAL AVERAGES OF OBSERVABLES

3.1. Matrix Elements of the Resolvent

In this section, we investigate the evolution of the average $(A, \hat{P}^t\rho)$ of an observable A with respect to an initial density ρ. First, we consider the Neumann series of the matrix element of the resolvent:

$$\left(A, \frac{1}{z\mathbf{1} - \hat{P}}\, \rho \right) = \sum_{t=0}^{+\infty} \frac{1}{z^{t+1}}\, (A, \hat{P}^t \rho) \tag{12}$$

where **1** is the identity operator. Each term of the series can be rewritten as

$$(A, \hat{P}^t\rho) = (A \circ \phi^t, \rho) = \sum_{k=0}^{\infty} \tilde{\rho}_k B_k(t) \tag{13}$$

where

$$B_k(t) = \begin{cases} \int_{\xi_{k-1}}^{\xi_k} dx\, A \circ \phi^t(x) & (k = 1, 2, \ldots) \\ \int_1^a dx\, A \circ \phi^t(x) & (k = 0) \end{cases} \tag{14}$$

From (1), one obtains the recursion relation of $B_k(t)$. For $k \geqslant 1$, it reads as

$$\begin{aligned} B_k(t+1) &= \int_{\xi_{k-1}}^{\xi_k} dx\, A \circ \phi^t(\eta_k(x-\xi_k)+\xi_{k-1}) \\ &= \frac{1}{\eta_k} \int_{\xi_{k-2}}^{\xi_{k-1}} dx\, A \circ \phi^t(x) = \frac{1}{\eta_k} B_{k-1}(t) \end{aligned} \tag{15}$$

and, hence, $\hat{B}_k(z) \equiv \sum_{t=0}^{\infty} B_k(t)/z^{t+1}$ satisfies

$$\hat{B}_k(z) = \frac{1}{z} B_k(0) + \frac{1}{z\eta_k} \hat{B}_{k-1}(z) \tag{16}$$

Similarly, one has

$$\hat{B}_0(z) = \frac{1}{z} B_0(0) + \frac{1-a}{z} \sum_{k=0}^{+\infty} \hat{B}_k(z) \tag{17}$$

The recursion relations (16) and (17) give

$$\hat{B}_0(z) = \frac{\Phi(z)}{Z(z)} \tag{18}$$

$$\hat{B}_k(z) = \frac{a\Phi(z)}{(1-a)\,\zeta(\beta)\,Z(z)} \frac{1}{z^k k^\beta} + \sum_{l=1}^{k} \frac{1}{z^{k-l+1}} \left(\frac{l}{k}\right)^\beta B_l(0) \quad (k \geqslant 1) \tag{19}$$

where the functions $\Phi(z)$ and $Z(z)$ are defined by:

$$\Phi(z) = B_0(0) + (1-a) \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \left(\frac{l}{k+l} \right)^{\beta} B_l(0) \tag{20}$$

$$Z(z) = z - 1 + \frac{a}{\zeta(\beta)} \sum_{k=1}^{\infty} \frac{1 - z^{-k}}{k^{\beta}} \tag{21}$$

Therefore, the matrix element of the resolvent of the Frobenius–Perron operator is:

$$\begin{aligned} \left(A, \frac{1}{z \mathbf{1} - \hat{P}} \rho \right) &= \sum_{k=0}^{\infty} \tilde{\rho}_k \hat{B}_k(z) \\ &= \frac{\Psi(z)\, \Phi(z)}{Z(z)} + \frac{(1-a)\, \zeta(\beta)}{a} \Bigg\{ B_1(0) [\Psi(z) - \tilde{\rho}_0] \\ &\quad + \sum_{l=2}^{\infty} l^{\beta} B_l(0)\, z^{l-1} \left[\Psi(z) - \tilde{\rho}_0 - \frac{a}{(1-a)\, \zeta(\beta)} \sum_{k=1}^{l-1} \frac{1}{z^k} \frac{\tilde{\rho}_k}{k^{\beta}} \right] \Bigg\} \end{aligned} \tag{22}$$

where the function $\Psi(z)$ is

$$\Psi(z) = \tilde{\rho}_0 + \frac{a}{(1-a)\, \zeta(\beta)} \sum_{k=1}^{\infty} \frac{\tilde{\rho}_k}{z^k k^{\beta}} \tag{23}$$

Infinite series in (20), (21), (22) and (23) are absolutely convergent for $|z| > 1$ and, thus, are analytic there. In addition, we note that

$$(A, \hat{P}^t \rho) = \oint_{|z|=r} \frac{dz}{2\pi i}\, z^t \left(A, \frac{1}{z \mathbf{1} - \hat{P}} \rho \right) \tag{24}$$

where the integration path is a counter-clockwise circle centered at $z = 0$ with radius $r > 1$.

3.2. Analytical Properties of Individual Functions

Here we derive analytical continuations of the functions $\Phi(z)$, $Z(z)$ and $\Psi(z)$ into the unit disk $|z| < 1$. With the aid of the formula

$$\frac{1}{k^{\beta}} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} ds\, s^{\beta-1} e^{-ks} \tag{25}$$

the series expression of Φ can be rewritten as

$$\Phi(z) = B_0(0) + (1-a) \sum_{l=1}^{\infty} \frac{l^\beta B_l(0)}{z\Gamma(\beta)} \int_0^\infty ds\, s^{\beta-1} e^{-ls} \sum_{k=0}^{\infty} \frac{1}{z^k} e^{-ks}$$

$$= B_0(0) + \frac{1-a}{\Gamma(\beta)} \sum_{l=1}^{\infty} l^\beta B_l(0) \int_0^\infty ds \frac{s^{\beta-1} e^{-ls}}{z - e^{-s}} \tag{26}$$

The above calculations are justified because the summation converges uniformly in s provided $|z| > 1$.

We observe that, as a result of an inequality for bounded observables A:

$$l^\beta \, |B_l(0)| \leqslant l^\beta (\xi_{l-1} - \xi_l) \sup_x \, |A(x)| = \frac{a \sup_x |A(x)|}{\zeta(\beta)}, \tag{27}$$

the estimate

$$\frac{1-a}{\Gamma(\beta)} \sum_{l=1}^{\infty} \left| l^\beta B_l(0) \int_0^\infty ds \frac{s^{\beta-1} e^{-ls}}{z - e^{-s}} \right| \leqslant \frac{a(1-a) \sup_x |A(x)|}{d(z, [0, 1])} \tag{28}$$

holds where $d(z, [0, 1])$ is the distance between z and the real interval $[0, 1]$. It is obvious that each term of the right-hand side of (26) is analytic except the cut on the real interval $[0, 1]$. Therefore, the right-hand side of (26) defines a function which is analytic except the cut on the real interval $[0, 1]$ and, then, is an analytical continuation of Φ from the outside of the unit disk to the whole complex plane.

Similarly, the following analytical continuations are obtained:

$$\Psi(z) = \tilde{\rho}_0 + \frac{a}{(1-a)\, \zeta(\beta)\, \Gamma[(\beta-1)\, l + \beta]} \sum_{l=0}^{\infty} \rho_l \int_0^\infty du \frac{u^{(\beta-1)(l+1)} e^{-u}}{z - e^{-u}} \tag{29}$$

$$Z(z) = (z-1) \left[1 + \frac{a}{\zeta(\beta)\, \Gamma(\beta)} \int_0^\infty ds \frac{s^{\beta-1}}{(e^s - 1)(z - e^{-s})} \right] \tag{30}$$

both of which are analytic in the unit disk $|z| < 1$ except the cut on the real interval $[0, 1]$. In addition, the last term of (22) has the following analytical continuation:

$$\Xi(z) \equiv B_1(0)\, \Psi(z) + \sum_{l=2}^{\infty} l^\beta B_l(0)\, z^{l-1} \left[\Psi(z) - \tilde{\rho}_0 - \frac{a}{(1-a)\, \zeta(\beta)} \sum_{k=1}^{l-1} \frac{1}{z^k} \frac{\tilde{\rho}_k}{k^\beta} \right]$$

$$\to \Psi(z) \sum_{l=1}^{\infty} l^\beta B_l(0)\, z^{l-1} - \sum_{l=2}^{\infty} l^\beta B_l(0)\, z^{l-1} \left[\tilde{\rho}_0 + \frac{a}{(1-a)\, \zeta(\beta)} \sum_{k=1}^{l-1} \frac{1}{z^k} \frac{\tilde{\rho}_k}{k^\beta} \right] \tag{31}$$

The series of the right-hand side are absolutely convergent for $|z|<1$ and, thus, analytic there. Note that (31) is not a simple rewrite as the left-hand side does not converge for $|z|<1$.

Next we consider the zeros of $Z(z)$ in the complement of the real interval $[0, 1]$. Let $\Omega(z) \equiv Z(z)/(z-1)$, then for $\operatorname{Im} z \neq 0$

$$\operatorname{Im} \Omega(z) = \frac{-a \operatorname{Im} z}{\zeta(\beta)\, \Gamma(\beta)} \int_0^\infty ds \frac{s^{\beta-1}}{(e^s-1)\, |z-e^{-s}|^2} \neq 0 \tag{32}$$

Also, when $\operatorname{Im} z = 0$ and $\operatorname{Re} z > 1$, we have $z - e^{-s} > 0$ and, thus,

$$\Omega(z) = 1 + \frac{a}{\zeta(\beta)\, \Gamma(\beta)} \int_0^\infty ds \frac{s^{\beta-1}}{(e^s-1)(z-e^{-s})} \geqslant 1 > 0 \tag{33}$$

On the other hand, for $\operatorname{Im} z = 0$ and $\operatorname{Re} z < 0$, we have

$$\Omega'(z) = \frac{-a}{\zeta(\beta)\, \Gamma(\beta)} \int_0^\infty ds \frac{s^{\beta-1}}{(e^s-1)(z-e^{-s})^2} < 0 \tag{34}$$

$$\Omega(-1) = 1 - a + \frac{a}{2^\beta} > 0 \tag{35}$$

and

$$\begin{aligned} \frac{\zeta(\beta)\, \Gamma(\beta)}{a} (\Omega(z)-1) &= -\int_0^\infty ds \frac{s^{\beta-1}}{(e^s-1)(|z|+e^{-s})} \\ &\leqslant -\int_1^\infty ds \frac{s^{\beta-1}}{(e^s-1)(|z|+e^{-s})} \leqslant -\int_1^\infty ds \frac{e^{-s}}{|z|+e^{-s}} \\ &= -\log\{(e|z|)^{-1}+1\} \to -\infty \qquad (z \to 0) \end{aligned} \tag{36}$$

Therefore, $\Omega(z)$ has a unique zero on the real interval $(-1, 0)$, which we denote $z = \lambda_d$.

3.3. Decomposition of Average Values of Observables

Now we go back to the contour integral (24). Because of the previous arguments, its integrand (i.e., z^t times the right-hand side of (22)) is analytic in $|z|<1$ except a simple pole at $z = \lambda_d$ and a cut on the real interval $[0, 1]$. The residue of the integrand at $z = \lambda_d$ is

$$\lim_{z \to \lambda_d} (z - \lambda_d)\, z^t \left(A, \frac{1}{z\mathbf{1} - \hat{P}} \rho \right) = \lambda_d^t \frac{\Psi(\lambda_d)\, \Phi(\lambda_d)}{(\lambda_d - 1)\, \Omega'(\lambda_d)} \tag{37}$$

Hence, the integration contour can be deformed from $\{z \mid |z| = r (r > 1)\}$ to

$$C \equiv \{z \mid |z| = \eta, |z-1| = \eta \text{ or } z = \lambda \pm i0 \; (\eta < \lambda < 1-\eta)\}$$

where $0 < \eta \ll 1$ and the direction of C is counter-clockwise. Then, because constants do not contribute to (24), one has

$$(A, \hat{P}^t \rho) = \lambda_d^t \frac{\Psi(\lambda_d)\, \Phi(\lambda_d)}{(\lambda_d - 1)\, \Omega'(\lambda_d)} + \int_C \frac{dz}{2\pi i} z^t \left\{ \frac{\Psi(z)\, \Phi(z)}{Z(z)} + \frac{(1-a)\, \zeta(\beta)}{a} \Xi(z) \right\} \tag{38}$$

Now we consider term by term. When A belongs to the functional space X_O and ρ to the space X_D, one has

$$\lim_{z \to 0} z z^t \frac{\Psi(z)\, \Phi(z)}{Z(z)} = 0 \tag{39}$$

and the function $z^\alpha \Xi(z)$ (with $\theta^{-1/(\beta-1)} < \alpha < 1$) is finite in the limit of $z \to 0$. Hence,

$$\lim_{\eta \to 0} \int_{|z| = \eta} \frac{dz}{2\pi i} z^t \left\{ \frac{\Psi(z)\, \Phi(z)}{Z(z)} + \frac{(1-a)\, \zeta(\beta)}{a} \Xi(z) \right\} = 0 \tag{40}$$

Similarly, under the same conditions, one has

$$\lim_{\eta \to 0} \int_{|z-1| = \eta} \frac{dz}{2\pi i} z^t \frac{\Psi(z)\, \Phi(z)}{Z(z)} = \lim_{z \to 1} (z-1) \frac{\Psi(z)\, \Phi(z)}{Z(z)} \tag{41}$$

$$\lim_{\eta \to 0} \int_{|z-1| = \eta} \frac{dz}{2\pi i} z^t \frac{(1-a)\, \zeta(\beta)}{a} \Xi(z) = 0 \tag{42}$$

To evaluate the limit in the right-hand side of (41), it is convenient to decompose

$$(z-1) \frac{\Phi(z)}{Z(z)} = (1-a)\, A(0) + (z-1) \frac{\Phi_r(z)}{Z(z)} \tag{43}$$

where $\Phi_r(z)$ is

$$\Phi_r(z) = \int_a^1 dx\{A(x) - A(0)\} + \frac{1-a}{\Gamma(\beta)} \sum_{l=1}^{\infty} l^{\beta} \int_{\xi_l}^{\xi_{l-1}} dx\{A(x) - A(0)\} \int_0^{\infty} ds\, \frac{s^{\beta-1} e^{-ls}}{z - e^{-s}} \tag{44}$$

Then, because $\Psi(z)$ is finite in the limit of $z \to 1$ and

$$\lim_{z \to 1} (z-1) \frac{\Phi_r(z)}{Z(z)} = \begin{cases} 0 & (\beta \leqslant 2) \\ \dfrac{\Phi_r(1)}{\Omega(1)} & (\beta > 2) \end{cases} \tag{45}$$

one has

$$\lim_{z \to 1} (z-1) \frac{\Psi(z)\, \Phi(z)}{Z(z)} = \begin{cases} \Psi(1)(1-a)\, A(0) & (\beta \leqslant 2) \\ \Psi(1)(1-a)\, A(0) + \dfrac{\Psi(1)\, \Phi_r(1)}{\Omega(1)} & (\beta > 2) \end{cases} \tag{46}$$

Note that, when $\beta \leqslant 2$, both $Z(z)$ and $\Phi_r(z)$ diverge as $z \to 1$, but the divergence of the former is stronger than that of the latter and the ratio $\Phi_r(z)/Z(z)$ vanishes for $z \to 1$.

Since $\Xi(z) = \Psi(z) \sum_{l=1}^{\infty} l^{\beta} B_l(0)\, z^{l-1} +$ (analytic term) for $|z| < 1$ (cf. (31)), the integral along the cut is evaluated as

$$\begin{aligned}
&\lim_{\eta \to 0} \int_{C \setminus \{|z| = \eta \cup |z-1| = \eta\}} \frac{dz}{2\pi i}\, z^t \left\{ \frac{\Psi(z)\, \Phi(z)}{Z(z)} + \frac{(1-a)\, \zeta(\beta)}{a} \Psi(z) \sum_{l=1}^{\infty} l^{\beta} B_l(0)\, z^{l-1} \right\} \\
&= \int_0^1 \frac{d\lambda}{2\pi i}\, \lambda^t \Bigg[-\left\{ \frac{\Psi(\lambda+i0)\, \Phi(\lambda+i0)}{Z(\lambda+i0)} + \frac{(1-a)\, \zeta(\beta)}{a} \Psi(\lambda+i0) \sum_{l=1}^{\infty} l^{\beta} B_l(0)\, \lambda^{l-1} \right\} \\
&\quad + \left\{ \frac{\Psi(\lambda-i0)\, \Phi(\lambda-i0)}{Z(\lambda-i0)} + \frac{(1-a)\, \zeta(\beta)}{a} \Psi(\lambda-i0) \sum_{l=1}^{\infty} l^{\beta} B_l(0)\, \lambda^{l-1} \right\} \Bigg] \\
&= \frac{a}{\zeta(\beta)\, \Gamma(\beta)} \int_0^1 d\lambda\, \frac{\mathrm{Im}\{\Psi(\lambda-i0)\, Z(\lambda+i0)\}}{\mathrm{Im}\, Z(\lambda+i0)} \\
&\quad \times \frac{\lambda^t\, (\log \frac{1}{\lambda})^{\beta-1}}{|Z(\lambda+i0)|^2} \frac{\mathrm{Im}\{\Phi(\lambda-i0)\, Z(\lambda+i0)\}}{\mathrm{Im}\, Z(\lambda+i0)}
\end{aligned} \tag{47}$$

where we have used

$$-\frac{\mathrm{Im}\,\Phi(\lambda \pm i0)}{\mathrm{Im}\, Z(\lambda \pm i0)} = \frac{(1-a)\,\zeta(\beta)}{a} \sum_{l=1}^{\infty} l^{\beta} B_l(0)\, \lambda^{l-1},$$

$$\mathrm{Im}\, Z(\lambda + i0) = \frac{\pi a}{\zeta(\beta)\, \Gamma(\beta)} \left(\log \frac{1}{\lambda}\right)^{\beta-1}.$$

With the aid of the following relation obtained from (43):

$$\mathrm{Im}\{\Phi(\lambda - i0)\, Z(\lambda + i0)\} = \mathrm{Im}\{\Phi_r(\lambda - i0)\, Z(\lambda + i0)\}, \tag{48}$$

Eqs. (38), (40), (42), (46) and (47) cast into

$$(A, \hat{P}^t \rho) = (A, F_{in})(\tilde{F}_{in}, \rho) + \lambda_d^t (A, F_d)(\tilde{F}_d, \rho) + \int_0^1 d\lambda\, \lambda^t (A, F_\lambda)(\tilde{F}_\lambda, \rho) \tag{49}$$

The linear functionals F_{in}, F_d and F_λ acting on the functions A are defined by

$$(A, F_{in}) = \begin{cases} \dfrac{1}{\hat{\Omega}(1)} \displaystyle\int_0^1 dx\, A(x)\, f_1(x) & (\beta > 2) \\ A(0) & (\beta \leqslant 2) \end{cases} \tag{50}$$

$$(A, F_d) = \frac{1}{(\lambda_d - 1)\, \hat{\Omega}'(\lambda_d)} \int_0^1 dx\, A(x)\, f_{\lambda_d}(x) \tag{51}$$

$$(A, F_\lambda) = N(\lambda) \int_0^1 dx \{A(x) - A(0)\} \times \left[f_\lambda(x) + \frac{\zeta(\beta)}{a} (\lambda - 1)\, \hat{\Omega}(\lambda) \sum_{l=1}^{\infty} l^{\beta} \chi_l(x)\, \lambda^{l-1} \right] \tag{52}$$

In the above, the functions $\chi_l(x)$, $f_\lambda(x)$, $\hat{\Omega}(\lambda)$ and $N(\lambda)$ are defined, respectively, as

$$\chi_l(x) = \begin{cases} 1 & (\xi_l \leqslant x < \xi_{l-1}) \\ 0 & (\text{otherwise}) \end{cases} \tag{53}$$

$$f_\lambda(x) = \frac{\chi_0(x)}{1-a} + \frac{1}{\Gamma(\beta)} \sum_{l=1}^{\infty} l^{\beta} \chi_l(x) \int_0^{\infty} ds\, \mathscr{P} \frac{s^{\beta-1} e^{-ls}}{\lambda - e^{-s}} \tag{54}$$

$$\hat{\Omega}(\lambda) = 1 + \frac{a}{\Gamma(\beta)\,\zeta(\beta)} \int_0^\infty ds\, \mathscr{P}\, \frac{s^{\beta-1}}{(e^s-1)(\lambda-e^{-s})} \tag{55}$$

$$N(\lambda) = \frac{\frac{a}{\Gamma(\beta)\,\zeta(\beta)}\,(\log\frac{1}{\lambda})^{\beta-1}}{(\lambda-1)^2\,\hat{\Omega}(\lambda)^2 + (\frac{a\pi}{\Gamma(\beta)\,\zeta(\beta)})^2\,(\log\frac{1}{\lambda})^{2\beta-2}} \tag{56}$$

where the convention $\xi_{-1} = 1$ is used and $\mathscr{P}$ stands for Cauchy's principal value.

On the other hand, the linear functionals $\tilde{F}_{in}$, $\tilde{F}_d$ and $\tilde{F}_\lambda$ acting on the initial densities ρ are given by

$$(\tilde{F}_{in}, \rho) = (1-a)\,\tilde{\rho}_0 + \sum_{l=0}^{\infty} \rho_l \nu_l(1) \tag{57}$$

$$(\tilde{F}_d, \rho) = (1-a)\,\tilde{\rho}_0 + \sum_{l=0}^{\infty} \rho_l \nu_l(\lambda_d) \tag{58}$$

$$(\tilde{F}_\lambda, \rho) = (1-a)\,\tilde{\rho}_0 + \sum_{l=0}^{\infty} \rho_l \left\{ \nu_l(\lambda) + \frac{(\lambda-1)\,\hat{\Omega}(\lambda)\,\Gamma(\beta)(\log\frac{1}{\lambda})^{(\beta-1)\,l}}{\Gamma[(\beta-1)\,l+\beta]} \right\} \tag{59}$$

where the coefficients $\nu_l(\lambda)$ are defined as

$$\nu_l(\lambda) = \frac{a}{\zeta(\beta)\,\Gamma[(\beta-1)\,l+\beta]} \int_0^\infty ds\, \mathscr{P}\, \frac{s^{(\beta-1)(l+1)}e^{-s}}{\lambda-e^{-s}} \tag{60}$$

3.4. Generalized Eigenfunctions of the Frobenius–Perron Operator

We remark that, for any bounded linear functional F over X_O, $(\hat{P}^*A, F)$ $(A \in X_O)$ is well defined due to the invariance of X_O under $\hat{P}^*$ and that the operator $\hat{P}$ can be extended via $(A, \hat{P}F) \equiv (\hat{P}^*A, F)$ to the space of all bounded linear functionals over X_O (the dual space of X_O). As seen by a straightforward calculation, the linear functionals F_{in}, F_d and F_λ $(0 < \lambda < 1)$ are bounded and satisfy

$$(A, \hat{P}F_{in}) \equiv (\hat{P}^*A, F_{in}) = (A, F_{in}), \tag{61}$$

$$(A, \hat{P}F_d) \equiv (\hat{P}^*A, F_d) = \lambda_d(A, F_d), \tag{62}$$

$$(A, \hat{P}F_\lambda) \equiv (\hat{P}^*A, F_\lambda) = \lambda(A, F_\lambda), \tag{63}$$

for any $A \in X_O$. Namely, they are eigenfunctions of $\hat{P}$ in the generalized sense.[12]

Similarly, the operator $\hat{P}^*$ can be extended to the dual space of X_D and the functionals $\tilde{F}_{in}$, $\tilde{F}_d$, $\tilde{F}_\lambda$ are its generalized eigenfunctions:

$$(\hat{P}^* \tilde{F}_{in}, \rho) = (\tilde{F}_{in}, \hat{P}\rho) = (\tilde{F}_{in}, \rho), \tag{64}$$

$$(\hat{P}^* \tilde{F}_{d}, \rho) = (\tilde{F}_{d}, \hat{P}\rho) = \lambda_d(\tilde{F}_{d}, \rho), \tag{65}$$

$$(\hat{P}^* \tilde{F}_{\lambda}, \rho) = (\tilde{F}_{\lambda}, \hat{P}\rho) = \lambda(\tilde{F}_{\lambda}, \rho). \tag{66}$$

Note that eigenfunctions of $\hat{P}$ and $\hat{P}^*$ are referred to, respectively, as right and left eigenfunctions of $\hat{P}$.

It is then interesting to reinterpret (49) from this point of view. One has

$$(A, \rho) = (A, F_{in})(\tilde{F}_{in}, \rho) + (A, F_d)(\tilde{F}_d, \rho) + \int_0^1 d\lambda (A, F_\lambda)(\tilde{F}_\lambda, \rho) \tag{67}$$

$$(A, \hat{P}\rho) = (A, F_{in})(\tilde{F}_{in}, \rho) + \lambda_d (A, F_d)(\tilde{F}_d, \rho) + \int_0^1 d\lambda\, \lambda (A, F_\lambda)(\tilde{F}_\lambda, \rho). \tag{68}$$

The first relation (67) corresponds to the completeness of the set of left eigenfunctions $\{F_{in}, F_d, F_\lambda\}$ and that of right ones $\{\tilde{F}_{in}, \tilde{F}_d, \tilde{F}_\lambda\}$. The second relation (68) can be considered as a spectral decomposition of $\hat{P}$ in terms of those eigenfunctions, which correspond to two isolated eigenvalues (1 and λ_d) and a continuous spectrum on $[0, 1]$. These relations are regarded as an extension to the Frobenius–Perron operator of the generalized spectral decomposition in the sense of ref. 12, which was originally formulated for self-adjoint and unitary operators. Such extended spectral decompositions are widely used to study statistical properties of dynamical systems.[11, 13]

4. LONG TIME BEHAVIOR

When $\lambda \simeq 1$, $(\tilde{F}_\lambda, \rho) \simeq (\tilde{F}_1, \rho) = (\tilde{F}_{in}, \rho)$ and the leading term of (A, F_λ) is

$$(A, F_\lambda) \simeq \begin{cases} K_1 (1-\lambda)^{\beta-3} \int_0^1 dx \{A(x) - A(0)\}\, f_1(x) & (\beta > 2) \\ \dfrac{K_2}{(1-\lambda)[\log(1-\lambda)]^2} \int_0^1 dx \{A(x) - A(0)\}\, f_1(x) & (\beta = 2) \\ K_3 (1-\lambda)^{1-\beta} \int_0^1 dx \{A(x) - A(0)\}\, f_1(x) & (2 > \beta > 3/2) \\ \dfrac{K_4}{\sqrt{1-\lambda}} \log \dfrac{1}{1-\lambda}\, A'(0) & (\beta = 3/2) \\ K_5 (1-\lambda)^{\beta-2}\, A'(0) & (\beta < 3/2) \end{cases} \tag{69}$$

where K_j $(j=1,\ldots,5)$ are positive constants. Those properties lead to the following behavior for $t \to +\infty$:

(i) *Stationary regime* $(\beta > 2)$

$$(A, \hat{P}^t \rho) \simeq (\tilde{F}_{in}, \rho) \left\{ \frac{1}{\hat{\Omega}(1)} \int_0^1 dx\, A(x)\, f_1(x) + \frac{K_1'}{t^{\beta-2}} \int_0^1 dx \{A(x) - A(0)\}\, f_1(x) \right\} \tag{70}$$

where $\hat{\Omega}(1) = \int_0^1 dx\, f_1(x)$ is the normalization constant and K_1' is a positive constant.

(ii) *Non-stationary regular regime* $(2 \geqslant \beta > 3/2)$

$$(A, \hat{P}^t \rho) \simeq (\tilde{F}_{in}, \rho) \left\{ A(0) + \frac{K_2'}{\log t} \int_0^1 dx \{A(x) - A(0)\}\, f_1(x) \right\}, \quad (\beta = 2) \tag{71}$$

$$(A, \hat{P}^t \rho) \simeq (\tilde{F}_{in}, \rho) \left\{ A(0) + \frac{K_3'}{t^{2-\beta}} \int_0^1 dx \{A(x) - A(0)\}\, f_1(x) \right\}, \quad (2 > \beta > 3/2) \tag{72}$$

where K_2' and K_3' are positive constants.

(iii) *Non-stationary singular regime* $(3/2 \geqslant \beta)$

$$(A, \hat{P}^t \rho) \simeq (\tilde{F}_{in}, \rho) \left\{ A(0) + \frac{K_4' \log t}{\sqrt{t}} A'(0) \right\}, \qquad (\beta = 3/2) \tag{73}$$

$$(A, \hat{P}^t \rho) \simeq (\tilde{F}_{in}, \rho) \left\{ A(0) + \frac{K_5'}{t^{\beta-1}} A'(0) \right\}, \qquad (3/2 > \beta) \tag{74}$$

where K_4' and K_5' are positive constants.

First we observe that the long-time behaviors in the stationary regime give the power-spectra discussed in ref. 5. At first sight, the behaviors in the non-stationary regime seem to be different from those of ref. 5. This is due to our choice (9) and (10) of initial densities. Indeed, if the function $\Psi(z)$ related to the initial densities has the same singularities near $z = 1$ as $\Phi(z)$, the power-spectra of ref. 5 are reproduced. But our choice leads to $\Psi(z)$ which is regular at $z = 1$.

Now we turn to the invariant function $f_1(x)$ defined by (54). For each value of $x \in (0, 1]$, the sum in (54) is finite and, thus, it is well-defined for any $\beta > 1$. As easily seen, it diverges at the origin as $f_1(x) \propto x^{-1/(\beta-1)}$. When $\hat{\Omega}(1) = \int_0^1 f_1(x)\, dx$ is finite, $f_1(x)/\hat{\Omega}(1)$ defines an invariant measure. This is the case for $\beta > 2$.

Provided that $A(x)$ is differentiable at $x=0$ and $A(x)-\{A(0)+A'(0)\,x\}=\mathrm{O}(x^{\beta/(\beta-1)})$ near $x=0$, the integral

$$\int_0^1 \{A(x)-A(0)\}\, f_1(x)\, dx \tag{75}$$

is finite for $\beta>3/2$. Then, since $\hat{\Omega}(1)\to+\infty$ ($\beta\to 2$), we have

$$\frac{1}{\hat{\Omega}(1)}\int_0^1 A(x)\, f_1(x)\, dx$$

$$= A(0)+\frac{1}{\hat{\Omega}(1)}\int_0^1 \{A(x)-A(0)\}\, f_1(x)\, dx \to A(0) \quad (\beta\to 2) \tag{76}$$

or the invariant measure $f_1(x)/\hat{\Omega}(1)$ weakly converges to $\delta(x)$ as $\beta\to 2$. This explains the transition from the stationary regime (70) to the non-stationary regime (71).

It is interesting that even in the non-stationary regime, the invariant function f_1 appears in the generalized eigenfunctions corresponding to the continuous spectrum (cf. (71) and (72)). In this way, non-normalizable measure in the non-stationary regime plays a role of a generalized eigenfunction. Moreover, because of the singularity of $f_1(x)$ at the origin, the integral (75) does not exist for $\beta\leqslant 3/2$ and, as a result, the coefficient of $1/t^{2-\beta}$ in (72) cannot keep the same form there. This explains the second transition from (72) to (73).

5. CONCLUSIONS

We have studied, for a piecewise linear approximation of Pomeau–Manneville map, the evolution of statistical averages of bounded observables which are differentiable at the origin with respect to piecewise constant initial densities which are smooth at the origin. The averages are shown to be a superposition of the contributions from two simple eigenvalues (1 and $\lambda_d\in(-1,0)$) and a continuous spectrum on $[0,1]$ of the Frobenius–Perron operator. The corresponding generalized eigenfunctions are explicitly constructed. The power-law decay of correlations is controlled by the continuous spectrum, particularly by its values near 1. The appearance of the continuous spectrum starting from 1 is the main difference from the hyperbolic systems, where correlation decays exponentially and decay rates are characterized as isolated eigenvalues of the Frobenius–Perron operator in a generalized sense.[1, 2, 11] Note that the space X_D of initial densities is not dense in $L^2[0,1]$ and we will discuss elsewhere a more satisfactory description in terms of initial densities which form a dense subset of $L^2[0, 1]$.

When the map is in the stationary regime, the map admits an invariant measure with non-zero density with respect to the Lebesgue measure. The invariant density is proportional to an invariant function f_1, which is meaningful for any β and depends smoothly on β. When the map is in non-stationary regime ($\beta \leqslant 2$), the invariant function $f_1(x)$ defines a non-normalizable measure. We show that this non-normalizable measure determines the generalized eigenfunction corresponding to the continuous spectrum at the spectral value 1; $F_\lambda|_{\lambda=1}$. Moreover, as the map approaches non-stationary regime (i.e., $\beta \to 2$), the normalizable invariant measure $f_1(x)/\hat{\Omega}(1)$ in the stationary regime is shown to converge weakly to a δ-function measure located at the marginal fixed point $x = 0$.

As shown in ref. 5, the power-spectrum changes its character in the non-stationary regime when the map ϕ behaves as $\phi(x) \simeq x + c_0 x^3$ near the marginal fixed point $x = 0$. The corresponding change of the power-law decay is observed in the present model with the threshold $\beta = 3/2$. Although we did not discuss in detail, the power-law decay above the threshold $\beta > 3/2$ is controlled by the cut of the denominator function $Z(z)$, which is essentially the dynamical zeta function. On the other hand, below the threshold $\beta \leqslant 3/2$, the power-law decay is controlled by the cut of the numerator function $\Phi(z)$. Thus the latter contribution cannot be obtained from the zeta function. Similar aspects in the power-spectrum was discussed by Hasegawa and Luschei.[7]

The real eigenvalue at $\lambda_d \in (-1, 0)$ corresponds to oscillatory exponential decay. Although it is not important in the long-time behavior, this was not known before.

Those considerations can be straightforwardly extended to the maps with anomalous diffusion such as the ones introduced by Geisel *et al.*,[9] which provide a spectral characterization of the anomalous diffusion. In order to study a nonlinear multibaker map of Gilbert, Ferguson, and Dorfman,[14] such an extension is necessary and it will be discussed elsewhere.

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